Limited liability constraints in adverse selection and moral hazard problems

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Abstract

This paper analyzes a principal-agent model in which the agent has private information before contracting with the principal and ex-post private decision. These inputs determine the random output accruing to the principal. We characterize the optimal menu of contracts between a risk-neutral principal and a risk-neutral agent when the agent has limited liability on his transfer or utility levels. We show that the lessons from the basic agency literature do not straightforwardly extend in this more complex environment. We highlight a key feature of these contracts: the agent faces endogenous countervailing incentives. The agent has an incentive to understate his efficiency due to the presence of adverse selection, and to overstate it for limited liability reasons.

Keywords Limited liability, Adverse selection, Moral hazard, Countervailing incentives

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1 Introduction

Most analyses of the incentives literature have investigated the implications of liability restrictions for contracts where only one type of private information is available to the agent.¹ This paper explores how limited liability constraints affect contractual terms when the probability of “success” depends on both the risk-neutral agent’s effort and ability, unobservable to the principal. The limited liability constraint is thus structured by the joint presence of adverse selection and moral hazard. We highlight a key feature of such limited liability models: the agent faces endogenous countervailing incentives.

The assumption of limited liability is founded upon several justifications ranging from wealth constraints or bankruptcy clauses to paternalism or equity considerations (Sappington, 1983). Limited liability constraints are common in labor contracting and in financial markets, to give a few examples. An employer is not free to punish poor performance with negative transfers; an entrepreneur cannot be asked to pay back more than he earns. More sophisticated limited liability constraints emerge naturally in these economic contexts. For example, it seems a reasonable supposition that the probability of default of a risky investment project is a function of both the entrepreneur’s ability to run the firm and his effort, unobservable to creditors. So the entrepreneur’s obligation to repay creditors can be conditioned upon both the stochastic output of the firm and the ability parameter. The optimal allocation will be structured by the presence of adverse selection before contracting, as will the liability restriction.

A stronger constraint is to assume that the agent cannot be forced to accept a profit or utility loss. The agent makes his move after observing his type but before observing the outcome of the random shock. Therefore, it may well be that the actual utility he gets, i.e. his utility after the realization of the state of nature, is negative despite his voluntary participation ex-ante. The limited liability constraint on the agent’s utility levels (or ex-post individual rationality constraint) rules out the possibility that the agent ends up with a negative utility levels after announcing his type and choosing his effort, for any realization

¹The liability constraint is based upon the realization of the stochastic output under moral hazard (e.g., Park, 1995), upon the agent’s type under adverse selection with ex-ante contracting (e.g., Sappington, 1983), or upon an ex-post observable signal correlated with agent’s true type (e.g., Demougin and Garvie, 1991).
of the output. It guarantees a minimum level of “well-being” to the agent. For example, an employer adding an incentive component to pay will also have to rise the base salary in order to cover the added disutility associated with greater effort work. The limited liability constraint on the utility levels involves both the sharing rule and the effort prescription. It is bivariate in terms of the domain on which the principal’s optimization problem is defined. It implies that the presence of adverse selection before contracting imposes structure on both variables.

The joint presence of adverse selection and moral hazard affects the basic trade-off between the extraction of a limited liability rent and allocative efficiency in a non-trivial way. It creates endogenous countervailing incentives to lie. The intuition is as follows. The agent has an incentive to understate his ability in order to convince the principal that a high performance is costly to achieve, and thus greater compensation is required. The principal mitigates this incentive by promising higher-powered incentives to higher types. But the presence of a liability restriction reduces the set of incentive feasible allocations. The principal will have to reward efficient types with extra limited liability rent if performance is high which, in turn makes the contract offered to efficient types attractive for inefficient types.

The effect of these conflicting incentives to lie on the optimal contract will depend on the strength of the interaction between the adverse selection and the moral hazard problems. The strength of the “coupling” between the two components can be summarized as follows:

- **Strong interaction:** If effort and ability enter as complements into the probability of success under a liability constraint on the utility levels, the principal cannot disentangle adverse selection from moral hazard. The dominating incentive is to overstate efficiency. The limited liability constraint on utilities in case of failure is decreasing for all types. It is binding for the most efficient type. The agent receives an information rent added up with the limited liability rent. The principal best reduces this information rent by expanding the optimal effort beyond its efficient level and thus mitigates the agent’s incentive to overstate type.

- **Weak interaction:** If effort and ability enter as complements into the probability of

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success under a liability constraint on the transfer payments, the moral hazard problem overcomes the adverse selection problem. The agent receives a limited liability rent but no informational rent. The pure moral hazard equilibrium is achieved despite the presence of adverse selection.\footnote{See assumption 1.}

- Medium interaction: If effort and ability enter as substitutes into the probability of success under a liability constraint on the transfers or on the utilities, the moral hazard problem overcomes the adverse selection problem since the agent receives a limited liability rent but no informational rent. But the pure moral hazard solution cannot be achieved since the adverse selection component plays a role through the sufficient condition for information revelation.

2 Related literature

Generalized agency We examine contractual design in a generalized principal-agent model, as defined by Myerson (1982). The principal has to make “honesty” and “obedience” rational equilibrium strategies for the agent. Much of the literature has analyzed mixed models in frameworks that can be viewed as pure adverse selection models.\footnote{We survey generalized agency models assuming a continuum of types and efforts. Note however that the literature has examined generalized agency models in the two-type case (see Laffont and Martimort, 2002). For instance, Jullien et al. (2007) investigate a two outcome/two type model of moral hazard with adverse selection on the agent’s risk-aversion.} Elements of both adverse selection and moral hazard can be found in Laffont and Tirole (1986). In their model, the output is a deterministic function of the agent’s effort and type, so their results are closely related to the results of models with pure adverse selection. Picard (1987) and Caillaud et al. (1992) examine a mixed model where the principal and the agent are risk-neutral and the agent’s liability is unlimited, so the moral hazard problem vanishes. With a risk-averse agent, Faynzilberg and Kumar (1997) give the conditions leading to the optimal contract.\footnote{They show that for a technology where the agent’s inputs and the random productivity shock are separable, the monotone likelihood ratio property of the moral hazard literature is sufficient for the monotonicity in}
ment into the probability of success, the type-dependent candidate contract does not satisfy generalized incentive compatibility. In contrast, we show that when the agent is protected by a limited liability constraint on utility levels, this production technology may lead to a type-dependent optimal contract that satisfies generalized incentive compatibility. To the best of our knowledge, all the generalized principal-agent models with a continuum of inputs share the result that the optimal contract involves pooling.

**Limited liability and generalized agency** The idea of limited liability in a generalized principal-agent framework is not new in the literature but the presence of countervailing incentives has not been studied. The framework closest to our own is Lewis and Sappington (2000, 2001), who assume that the probability of success is influenced by the agent’s effort and by a productivity parameter, under conditions of limited liability on a risk-neutral agent transfers. Assuming that failure produces no value for the principal, the associated limited liability constraint is binding for all types. Instead, we assume a positive output in case of failure which allows the principal to potentially pay an information rent to the agent on top of the limited liability rent. Moreover, Lewis and Sappington (2000, 2001) assume that the probability of success is separable in effort and type, that effort is essential for success (complements) and the disutility for effort is linear.\(^6\) We show that under these assumptions and a liability restriction on the transfer payments to the agent, the pure moral hazard equilibrium is achieved despite the presence of adverse selection. An equivalent formulation of our model is to consider the disutility the agent incurs when he implements a given success probability in the particular state of nature that prevails. This formulation is close in spirit to Hiriart and Martimort (2006). In their model, the probability of “success” depends on the agent’s effort alone. Exerting effort implies a disutility for the agent. The agent also bears a production cost, and his type determines his potential for reducing production costs.\(^7\)

\(^6\)Lewis and Sappington use this setting: (2000) to analyze how wealth constraint affects optimal mechanisms for selling a project to bidders, (2001) to incorporate asymmetric knowledge of wealth.

\(^7\)See also Laffont (1995).
Countervailing incentives  The countervailing incentives discussed in this article differ substantially from those identified in the literature. While the source of countervailing incentives is exogenous in the literature, we provide an alternative framework where the reversal of incentives is endogenous. The analysis of countervailing incentives has been pioneered by Lewis and Sappington (1989a), where countervailing incentives arise due to inversely related variable and fixed costs. A similar effect occurs when the agent’s reservation utility is type dependent. Early contributions are for instance by Lewis and Sappington (1989b) and Maggi and Rodríguez-Clare (1995a) (see also Jullien, 2000, for a complete analysis). In these papers, countervailing incentives arise because outside the contractual relationship with the principal, efficient agents have better opportunities than inefficient agents. Unlike in standard theory, countervailing incentives emerge in our paper because of the presence of a limited liability constraint, that is a better opportunity for efficient agents within the contractual relationship with the principal.

The article proceeds as follows. We present the model in section 3. We then study the properties of the optimal contract in section 4. Section 5 concludes by discussing the consequences of imposing a limited liability constraint on utility rather than on the transfer payments. All the proofs are given in an appendix.

3 The Model

The risk-neutral principal owns a productive technology that requires as inputs the effort, $e$, and the productivity parameter, $\theta$, of the risk-neutral agent protected by a limited liability constraint. This effort and this type, together with the realization of a random productivity shock, $\varepsilon$, determine the value of output produced, $x$, according to the relationship $x = X(e, \theta, \varepsilon)$. We take the case where the output space is binary, $x \in X = \{x, \overline{x}\}$, whereas $e \in E = [\underline{e}, \overline{e}]$ and $\theta \in \Theta = [\underline{\theta}, \overline{\theta}]$.

8Countervailing incentives can arise also in the case of costly state falsification where the agent undertakes a costly action to distort the signal of marginal cost observed by the principal (Maggi and Rodriguez-Clare, 1995b).
**Information.** The principal observes the amount produced $x$ but not the agent’s level of effort and productivity parameter. The agent knows his type before signing the contract. The principal’s beliefs about the agent’s type are given by a density function $h(\theta)$ with $h(\theta) > 0$ for all $\theta \in \Theta$ and a distribution function $H(\theta)$. We assume the monotone hazard rate property $(d/d\theta)((H(\theta))/h(\theta)) \geq 0$ and $(d/d\theta)((H(\theta) - 1)/h(\theta)) \geq 0$, $\forall \theta \in \Theta$. The conditional density $\rho = f(e, \theta)$ indicates the likelihood of a realization $x$ given the inputs $(e, \theta)$. Let subscripts $i$ denote partial derivatives with the $i$th argument. We assume that greater effort increases the conditional probability of a realization $x$ at a decreasing rate, so $f_1(e, \theta) > 0$, and $f_{11}(e, \theta) \leq 0$. Hence the production technology satisfies the Mirrlees-Rogerson conditions. Moreover, higher type increases the conditional probability of a realization $x$ at a decreasing rate, i.e. $f_2(e, \theta) > 0$ and $f_{22}(e, \theta) \leq 0$ and the marginal productivity of effort, i.e. $f_{12}(e, \theta) \geq 0$. We assume that type and effort complement each other to produce $x$, but $\rho$ allows type and effort to be substitutes ($f_{12}(e, \theta) = 0$) or complements ($f_{12}(e, \theta) > 0$).

**Compensation.** The agent’s reward is made to depend on the amount produced and on the agent’s announcement of his type. It is specified by an upfront fee $a$ and a bonus $b$ being paid by the principal to the agent in case of a realization $x$.

**Indirect disutility function** $\psi(\rho, \theta)$. The agent is effort averse with $\varphi(e)$ his disutility for effort. The function $\varphi(e)$ satisfies the conditions $\varphi'(e) > 0$ and $\varphi''(e) > 0$. Instead of insisting on the agent choosing a level of effort we may view the agent as making his decision in terms of the level of the probability of success.\footnote{Indeed, effort directly lowers the agent’s satisfaction by the disutility associated with effort. Effort indirectly raises the agent’s utility by increasing the success probability and so the reward. See Hurwicz and Shapiro (1978) and Sappington (1983) among others.} With $e = g(\rho, \theta)$, we define $\psi(\rho, \theta) = \varphi(g(\rho, \theta))$ to be the disutility or personal cost incurred by the agent when conditional success probability $\rho$ is generated in state $\theta$. The properties of $\psi(\rho, \theta)$ follow from the assumptions above (see appendix A). In more productive states, the agent’s disutility of effort is smaller, i.e. $\psi_2(\rho, \theta) < 0$. Higher values of $\theta$ correspond to states in which a higher probability of success is less onerous to produce, i.e. $\psi_{12}(\rho, \theta) < 0$. For all types, the marginal disutility of effort
to the agent is positive and increasing, i.e. \( \psi_1(\rho, \theta) > 0 \) and \( \psi_{11}(\rho, \theta) > 0 \). Finally, the assumption that higher types increase the marginal productivity of effort implies that
\[
\frac{\psi_2(\rho, \theta)}{\psi_1(\rho, \theta)} \geq \frac{\psi_{12}(\rho, \theta)}{\psi_{11}(\rho, \theta)},
\]
with equality if \( f_{12}(e, \theta) = 0 \).

The principal’s objective function. In order to get information from the agent and influence his decision, the principal must design a generalized incentive feasible contract. She chooses \( a, b \) and \( \rho \) which maximize her expected gain:
\[
\int_{\Theta} (\rho (\bar{x} - a - b) + (1 - \rho) (x - a)) f(\theta) d\theta,
\]
subject to the generalized incentive compatible constraint, the participation constraint and the limited liability constraint.

Generalized incentive compatibility. Following Myerson (1982), the contract choice \((a(.), b(.), p(.))\), with \( a : \Theta \to A, b : \Theta \to B, \) and \( p : \Theta \to [0, 1] \) is generalized incentive compatible if it induces a truthful revelation of type and obedience from the agent. It will depend on the agent’s possible responses to any given reward formula. The agent of type \( \theta \) announces his type as \( \hat{\theta} \) and selects the conditional probability denoted by \( p(\hat{\theta}, \theta) \) such that, \( \forall \hat{\theta}, \theta \in \Theta, \forall \rho \in [0, 1] \)
\[
p(\hat{\theta}, \theta) = \arg \max_{\rho} \left\{ a(\hat{\theta}) + \rho b(\hat{\theta}) - \psi(\rho, \theta) \right\}.
\]
Abusing notation, if \( \hat{\theta} = \theta \), this probability is denoted by \( p(\theta, \theta) = p(\theta) \). The agent’s indirect expected utility function is \( U(\hat{\theta}, \theta) \) such that \( U(\hat{\theta}, \theta) = a(\hat{\theta}) + p(\hat{\theta}, \theta)b(\hat{\theta}) - \psi(p(\hat{\theta}, \theta), \theta) \). In particular, when the contract choice is generalized incentive compatible, the agent’s indirect expected utility is \( V(\theta) \) with
\[
V(\theta) = U(\hat{\theta}, \theta) = \max_{\hat{\theta} \in \Theta} U(\hat{\theta}, \theta) = a(\theta) + p(\theta)b(\theta) - \psi(p(\theta), \theta).
\]
Hence, the generalized incentive constraint is \( \forall \hat{\theta}, \theta \in \Theta, \forall \rho \in [0, 1] \)
\[
V(\theta) \geq a(\theta) + \rho b(\theta) - \psi(\rho, \theta).
\]
This constraint means that the agent with type $\theta$ prefers to reveal his type to the principal and obey his recommendation on what the level of conditional success probability in state $\theta$ should be.

Lemma 1 characterizes the necessary and sufficient conditions for (GIC).

**Lemma 1.** The allocation $(a, b, p)$ is generalized incentive compatible if and only if, $\forall \theta \in \Theta$

\[
\begin{align*}
 b(\theta) &= \psi_1(p(\theta), \theta) \\
 V'(\theta) &= -\psi_2(p(\theta), \theta) \\
 p'(\theta) &\geq -\frac{\psi_{12}(p(\theta), \theta)}{\psi_{11}(p(\theta), \theta)}. 
\end{align*}
\]

The conditional probability that maximizes the agent’s expected utility is given by (5). Equation (6) expresses truthful revelation of type and inequality (7) ensures sufficiency.

**Feasibility. Participation.** The agent will agree to the contract initially if it promises him an expected utility in excess of his reservation level, normalized to zero, i.e., $\forall \theta \in \Theta$

\[
V(\theta) \geq 0.
\]

**Limited liability.** We examine two forms of limited liability: nonnegativity constraint on the transfers to, and the ex-post utilities of, the agent. Let $\alpha$ be a dummy variable that is one for a liability restriction on the agent’s utility levels and zero for a liability restriction on the transfer payments to the agent. With $\alpha = 1$ (resp. $\alpha = 0$), there is a minimum utility (resp. wage), normalized to zero, that the agent receives provided he abides by the terms of the contract he signs, i.e., $\forall \theta \in \Theta$

\[
\begin{align*}
 l_1(\theta) &= a(\theta) - \alpha \psi(p(\theta), \theta) \geq 0, \\
 l(\theta) &= a(\theta) + b(\theta) - \alpha \psi(p(\theta), \theta) \geq 0
\end{align*}
\]

in case of a realization $\underline{x}$ and $\overline{x}$ respectively. Since $\psi_1(p(\theta), \theta) > 0$, it is straightforward with (5) that only the limited liability constraint at the lower output state is relevant, so (LL) boils down to $l_1(\theta) \geq 0$, or, with (4) and (5), to, $\forall \theta \in \Theta$

\[
l(\theta) = V(\theta) - p(\theta)\psi_1(p(\theta), \theta) + \psi(p(\theta), \theta)(1 - \alpha) \geq 0
\]
We assume that $\psi_1(\rho, \theta) \rho / \psi(\rho, \theta) \geq 1$, \(\forall \theta \in \Theta\), \(\forall \rho \in [0,1]\), so the relevant feasibility constraint is (8). In the sequel, we call $l(\theta)$ the agent’s net expected utility, that is the agent’s expected utility $(V(\theta))$ in excess of the limited liability rent $(p(\theta)\psi_1(p(\theta), \theta) - \psi(p(\theta), \theta)(1 - \alpha))$.

**The principal’s problem.** The principal seeks to maximize her expected gain (2) subject to (GIC) and (LL) encapsulated in expressions (6) to (8). With (6), we have

$$l'(\theta) = -\alpha \psi_2(p(\theta), \theta) - p(\theta)\psi_1(p(\theta), \theta) - p'(\theta)(\alpha \psi_1(p(\theta), \theta) + p(\theta)\psi_1(p(\theta), \theta)).$$  

(9)

Hence, with (5) and $l(\theta)$ in (LL), the principal’s problem reduces to the following

$$\max_{l(\theta), p(\theta)} \int_{\theta} \{p(\theta)(x - \bar{x}) + x - l(\theta) - p(\theta)\psi_1(p(\theta), \theta) - \alpha \psi(p(\theta), \theta)\} \ h(\theta) \ d\theta$$

s.t. (7), (8) and (9).

**Benchmark solutions.** Before characterizing the solution to the principal’s problem $[P]$, it is instructive to analyze the solutions to three benchmarks. We label as $[P-FB]$ the benchmark that gives the efficient probability of success, $p^*(\theta)$. It is obtained by equating the agent’s marginal disutility from generating a greater probability of success and the principal’s valuation of such probability of success, i.e. \(\forall \theta \in \Theta\)

$$\bar{x} - \bar{x} = \psi_1(p^*(\theta), \theta).$$  

(FB)

When the principal cannot observe the agent’s level of effort, the first-best probability of success is still implemented. The principal chooses incentive compatible transfers (5) which make (IR) binding for all $\theta \in \Theta$ and the agent is punished for failure, $l^*(\theta) = \psi(p^*(\theta), \theta)(1 - \alpha) - p^*(\theta)\psi_1(p^*(\theta), \theta) < 0$, \(\forall \theta \in \Theta\), \(\forall \alpha \in \{0,1\}\). The second benchmark $[P-AS]$ is one in which the agent’s ability is no longer observed by the principal. Moral hazard is not an issue despite the non-observability of effort because of the agent’s risk-neutrality. The principal

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10The disutility $\psi(\rho, \theta)$ is assumed to be elastic with respect to the probability of success $\rho$. Note that this assumption follows from the mean value theorem if $\psi(0, \theta) = 0$, \(\forall \theta \in \Theta\).
chooses a generalized incentive compatible contract (5) to (7) and \((IR)\) is binding at \(\theta\). The optimal probability of success \(p^{AS}(\theta)\) such that

\[
\bar{x} - \bar{x} = \psi_1(p^{AS}(\theta), \theta) + \frac{H(\theta) - 1}{h(\theta)}\psi_{12}(p^{AS}(\theta), \theta)
\]  

(AS)

is distorted downward except for the most efficient type \(\bar{\theta}\), and \(l^{AS}(\theta) < 0, \forall \theta \in \Theta, \forall \alpha \in \{0, 1\}\). Finally, consider the setting where the principal faces limited liability rules \((LL)\), observes the agent’s ability but cannot observe the agent’s effort. Call this third benchmark \([P-MH]\). In this problem, \((FB)\) is no longer implemented. The principal chooses the incentive compatible contract (5) which makes (8) binding for all \(\theta \in \Theta\). The optimal probability of success \(p^{MH}(\theta)\) such that

\[
\bar{x} - \bar{x} = \psi_1(p^{MH}(\theta), \theta)(1 + \alpha) + p^{MH}(\theta)\psi_{11}(p^{MH}(\theta), \theta)
\]  

(MH)

is distorted downward.

Section 4 discusses the properties of the solution to \([P]\). We show that the qualitative properties of the solution to the generalized agency model are significantly different from those arising in the basic agency models.

4 Properties of the solution to \([P]\)

As a preliminary step in solving \([P]\) we present a property that must hold in any generalized incentive feasible contract.

**Lemma 2.** The optimal contract specifies \(l'(\theta) \leq 0, \forall \theta \in \Theta\).

Lemma 2 states that, at the optimal contract, the agent will get a lower net expected utility level, the higher his type. The intuition for this is as follows. Assume unlimited liability and moral hazard only \((FB)\). The agent is “incentivized” by being rewarded for success and penalized for failure. But higher values of \(\theta\) are more productive states, so higher types are more likely to get the compensation associated with \(\bar{x}\). Hence, the principal provides additional incentives for lower types by promising \(a'(\theta) < 0\) and \(a'(\theta) + b'(\theta) < 0\). Suppose now that \(\theta\) is the agent’s private information. The agent has an incentive to
understate his type in order to convince the principal that a realization $\pi$ is very costly to achieve. To induce high types to select the contract designed for them, the contract offered to low types must be made sufficiently unattractive to high types. The principal designs the compensations $a(\theta) < 0$ and $a(\theta) + b(\theta) > 0$ to ensure that the wedge between those two levels induces truthful revelation of type, i.e.

$$a'(\theta) \leq 0 \text{ and } a'(\theta) + b'(\theta) \geq 0. \quad (10)$$

Expressions (10), together with (6) and (7) constrain the required slope of $l(\theta)$ to prevent such understatement (see equation (9)). Hence it is endogenously determined that the agent’s expected utility $V(\theta)$ and his limited liability rent $(p(\theta)b(\theta) - \psi(p(\theta), \theta)(1 - \alpha))$ will be increasing. This turns out to have particularly important implications for the solution to $[P]$. Because the limited liability rent is more attractive to high types, the agent has an incentive to overstate his type. The contract exhibits endogenous countervailing incentives: while ensuring that high types report the truth, the principal encourages the agent to overstate his type.

Let us provide an intuitive discussion of this countervailing incentives effect due to the presence of a limited liability constraint.\textsuperscript{11} We denote by $l_1(\hat{\theta}, \theta)$ the agent’s net expected utility level if the agent reports $\hat{\theta}$ when his type is really $\theta$. Since the truth is an optimal response for all $\theta$, it must be the case that $l_1(\theta, \theta) = U(\theta, \theta) - p(\theta, \theta)b(\theta) + \psi(p(\theta, \theta), \theta)(1 - \alpha))$. Lemma 2 is then equivalent to $l_1(\theta, \theta) + l_2(\theta, \theta) \leq 0$. With (1), (3), (B.1) and (B.2) in appendix B, $l_1(\theta, \theta)$ and $l_2(\theta, \theta)$ are endogenously determined such that

$$l_1(\theta, \theta) = -p_1(\theta, \theta)(\alpha \psi_1(p(\theta, \theta), \theta) + p(\theta, \theta)\psi_{11}(p(\theta, \theta), \theta)). \quad (11)$$

$$l_2(\theta, \theta) = \alpha \left( -\psi_2(p(\theta, \theta), \theta) + \frac{\psi_{12}(p(\theta, \theta), \theta)}{\psi_{11}(p(\theta, \theta), \theta)} \psi_1(p(\theta, \theta), \theta) \right) \leq 0. \quad (12)$$

The implications of the countervailing incentives depend critically on the shape of $l_2(\theta, \theta)$ and the principal adjusts $l_1(\theta, \theta)$ accordingly. More precisely, if $l_2(\theta, \theta) = 0 (< 0)$ the countervailing incentives are exactly offsetting (the agent has an incentive to overstate his type). Notice with (12) that $l_2(\theta, \theta) = 0 (< 0)$ if and only if $f_{12}(e, \theta) = 0$, or $\alpha = 0$, or both ($f_{12}(e, \theta) > 0$ and $\alpha = 1$). We choose to split the study according to the form of the probability of success.

\textsuperscript{11}The results will be derived formally in propositions 1 to 3.
In the sequel, superscripts $s$ and $c$ will refer respectively to the cases where effort and type enter as substitutes and complements into the probability of success. Subscripts $t$ and $u$ will denote respectively a framework in which the agent is protected by a limited liability constraint on the transfer payments ($\alpha = 0$) and on the utility levels ($\alpha = 1$). The results are illustrated in figures 1 to 3.\textsuperscript{12}

4.1 Effort and type are substitutes: $f_{12}(e, \theta) = 0$

If effort and type enter into the conditional success probability as substitutes, the marginal productivity of effort does not vary over type: the disutility of generating conditional probability of success $\rho$ in state $\theta$, $\psi(\rho, \theta)$, is a quasi-linear function. The marginal rate of trade-off between $\theta$ and $\rho$ for $\psi(\rho, \theta)$, $d\rho/d\theta = -\psi_2/\psi_1 = -\psi_{12}/\psi_{11} = f_2(e, \theta)$ depends only on $\theta$, not on $\rho$. In other words, there is no “effort effect” of a higher $\theta$ on the variation of $\rho$.

It follows that the expression between brackets in (12) is zero, so $L_2(\theta, \theta) = 0$ and the countervailing incentives are exactly offsetting. Then, since there is no reason for the principal to give more to the agent than is strictly necessary, it is obvious that the principal will set equation (11) equal to zero and the limited liability constraint is binding. The principal finds that it is no longer optimal to keep a wedge between $a(\theta)$ and $a(\theta) + b(\theta)$ that is increasing in types. Constraint (8) is binding on $\Theta$. The agent receives a limited liability rent but no rent from his private information: the moral hazard problem overcomes the adverse selection problem. All types in $\Theta$ are offered the same compensations.

The conditional probability of success is thus distorted downward with respect to the first-best in order to reduce the limited liability rent. However, it does not correspond exactly to the pure moral hazard solution ($MH$) because the presence of adverse selection plays a role through the sufficient condition. It follows that the optimal level of effort corresponds to the expected value of the optimal effort without adverse selection since the principal relies on her prior beliefs about $\theta$ to satisfy (7).

Proposition 1 summarizes the structure of the optimal contract when effort and types are substitutes.

\textsuperscript{12}In figures 1 to 3, we assume $\bar{x} - \underline{x} = 1$. 
Proposition 1. Assume $f_{12}(e, \theta) = 0$. At the optimal contract, constraint (8) is binding on $\theta \in \Theta$ and the optimal conditional success probability $p^*(\theta)$ is such that

(i) $p^*(\theta) = f(e^*, \theta)$, with $e^*$ such that
\[
\bar{x} - \bar{x} = \psi_1(f(e^*, \theta), \theta)(1 + \alpha) + \int_{\Theta} \psi_1(f(e^*, \theta)) \psi_1(f(e^*, \theta), \theta) h(\theta) d\theta
\]

(ii) $\forall \theta \in \Theta$, $p^*(\theta) < p^*(\theta)$.

The moral hazard problem overcomes the adverse selection problem since the agent receive a limited liability rent but no rent from his ex-ante private information. But even if the presence of adverse selection plays a secondary role in the model, it still has an impact on the optimal solution through the sufficient condition. Hence, we define the strength of the interaction between the adverse selection and the moral hazard problems as medium when $f_{12}(e, \theta) = 0$.

To illustrate our results, we consider the following parameterization of $f$: $f(e, \theta) = e + \theta$, with $E = [0, 0.5]$ and $\Theta = [0, 0.5]$. We assume that $\theta$ follows a uniform distribution over $\Theta$ and $\varphi(e) = e^2$. We get the cost incurred by the agent when probability $\rho$ is generated in state $\theta$ such that $\psi(\rho, \theta) = (\rho - \theta)^2$. The limited liability constraint is binding at the optimal contract and so the optimal sharing rule is type-independent. The optimal effort under generalized agency corresponds to the expected value of the optimal effort without adverse selection. Hence, we get $p^*(\theta) = \theta + \frac{\bar{x} - \bar{x}}{2(2+\alpha)} - \frac{1}{4(2+\alpha)}$. The restriction on utilities ($\alpha = 1$) yields a lower optimal probability of success than the restriction on transfers ($\alpha = 0$), as in the pure moral hazard framework. Indeed, when $f_{12}(e, \theta) = 0$, the presence of adverse selection does not modify the trade-off that prevails under moral hazard alone: introducing a stronger limited liability rule generates a larger distortion in the probability of success. The results are illustrated in figure 1.

4.2 Effort and types are complements: $f_{12}(e, \theta) > 0$

If effort and type enter into the conditional success probability as complements, a higher type strictly increases the marginal productivity of effort in terms of likelihood of a realization $\bar{x}$.
Figure 1: Probabilities of success at the solutions to [P-FB], [P-MH], [P-AS] and [P] when $f_{12}(e, \theta) = 0$. 
Under this set of production technology parameterizations, effort matters in explaining the variation of $\rho$ due to a higher $\theta$. However, the implications of the complementarity between effort and type on the optimal contact vary drastically according to the form of the limited liability constraint.

**Limited liability constraint on the transfer payments**: $\alpha = 0$. In this case, it is obvious from (12) that $L_2(\theta, \theta) = 0$ since $\alpha = 0$. This result follows directly from the fact that $a(\theta)$ does not depend on the agent’s true type. Hence, the interaction between adverse selection is such that the countervailing incentives offset each other, and so the analysis is analogous to the one before proposition 1. However, there appears to be an interesting difference between the optimal contracts when one makes the following assumption.

**Assumption 1.** $\psi_{12}(\rho, \theta)/\psi_{11}(\rho, \theta)$ is unit elastic with respect to $\rho$.

This assumption is satisfied under the standard Cobb-Douglas specification of the probability of success that captures the complementarity between effort and ability, and the power disutility from effort. In this case, the pure moral hazard equilibrium ($MH$) is achieved despite the agent’s ex-ante private information. The adverse selection problem vanishes completely since the sufficient condition (7) holds.

Proposition 2 presents the structure of the optimal contract in this case.

**Proposition 2.** Assume $f_{12}(e, \theta) > 0$, $\alpha = 0$ and assumption 1. At the optimal contract, constraint (8) is binding on $\theta \in \Theta$. The optimal conditional success probability $p^*_t(\theta)$ is such that

(i)

$$\bar{x} - \underline{x} = \psi_1(p^*_t(\theta), \theta) + p^*_t(\theta)\psi_{11}(p^*_t(\theta), \theta).$$

(ii)

$$p^*_t(\theta) = p^*_{MH}(\theta) < p^*(\theta)$$
The moral hazard problem overcomes the adverse selection problem since the agent receives a limited liability rent but no informational rent. But contrary to the medium interaction case, the sufficient condition (7) holds. Hence, we define the strength of the interaction between the adverse selection and moral hazard problems as weak when \( f_{12}(e, \theta) > 0, \alpha = 0 \) under assumption 1.

**Limited liability on the utility levels:** \( \alpha = 1 \). In this case, \( l_2(\theta, \theta) < 0 \) because the expression between brackets in (12) is negative since \( \alpha = 1 \) and since \( f_{12}(e, \theta) > 0 \) implies that \( \psi_{12}(\rho, \theta)/\psi_{11}(\rho, \theta) < \psi_2(\rho, \theta)/\psi_1(\rho, \theta) \). The dominating incentive is then to overstate efficiency. Therefore, the limited liability constraint is satisfied for all types if we set \( l(\tilde{\theta}) = 0 \). To mitigate the incentive to exaggerate \( \theta \), the principal will choose to make the slope of \( l(\theta) \) even more negative with \( l_1(\theta, \theta) < 0 \) in (11) by expending the conditional success probability beyond its efficient level. Indeed, higher conditional probability of success is more costly to generate when type is truly low. It reduces the potential gain from overstating \( \theta \). Hence, the optimal conditional probability level is distorted upward except at \( \tilde{\theta} \), higher powered incentives are provided to the agent, i.e. \( b'(\theta) > 0 \), so the optimal contract is fully separating.

The properties of the solution in this case are summarized in proposition 3.

**Proposition 3.** Assume \( f_{12}(e, \theta) > 0 \) and \( \alpha = 1 \). At the optimal contract, constraint (8) is binding only at \( \tilde{\theta} \). The optimal conditional success probability \( p_u^c(\theta) \) is such that

(i) \[
\bar{x} - \underline{x} = \psi_1(p_u^c(\theta), \theta) + \frac{H(\theta)}{h(\theta)} \psi_{12}(p_u^c(\theta), \theta)
\]

(ii) \[
p_u^c(\theta) \geq p^*(\theta)
\]

with equality holding at \( \tilde{\theta} \).

The agent receives an informational rent added up with the limited liability rent. So the interaction between the two components of the model is strong when \( f_{12}(e, \theta) > 0 \) and \( \alpha = 1 \).
Figure 2: Probabilities of success at the solutions to [P-FB], [P-MH], [P-AS] and [P] when $f_{12}(e, \theta) > 0$.

Consider the following example: $f(e, \theta) = e^{2/3} \theta^{1/3}$, with $E = [0, 1]$ and $\Theta = [0.5, 1]$. Assume that $\theta$ follows a uniform distribution over $\Theta$ and $\varphi(e) = e^2$. The cost incurred by the agent when probability $\rho$ is generated in state $\theta$ is $\psi(\rho, \theta) = \rho^3 / \theta$. From proposition 2, we get $p_{ct}(\theta) = \frac{1}{3} \sqrt{\theta (x - x)} = p_{ct}^{MH}(\theta)$. On the contrary, we found that the limited liability constraint on utilities generates an upward distortion in effort in the generalized agency setting. Hence, with proposition 3, we get $p_{cu}(\theta) = \theta \sqrt{\frac{2}{3} (x - x)}$. Constraint (GIC) is satisfied since condition (7) holds at $p_{cu}(\theta)$. All types, except $\theta = 1/2$, generate a conditional success probability higher than the first-best level, i.e. $p_{cu}(\theta) \geq p^*(\theta) = \sqrt{\frac{8}{3} (x - x)}$. The results are illustrated in figure 2.

In the next section, we analyze the consequences of imposing a limited liability constraint on utilities rather than a liability restriction on the transfer payments to the agent.
5 Limited liability on utilities versus limited liability on transfers

In an adverse selection model with interim individual rationality, it is obvious that the two constraints yield an equivalent problem for the principal. In this case, the limited liability constraint on utilities is equivalent to the individual rationality constraint and implies limited liability on transfers. This result does not hold in a moral hazard framework: the restriction on the utilities is a stricter limited liability rule so it generates a larger distortion in effort. Indeed, to increase $p(\theta)$, the principal has to raise $b(\theta)$. Raising $b(\theta)$ alone allows one to meet a limited liability constraint on transfers but does not preserve a liability restriction on utilities since a higher $p(\theta)$ increases $\psi(p(\theta), \theta)$. To meet the limited liability constraint on utilities, the principal must also increase $a(\theta)$, so the principal chooses to generate a lower $p(\theta)$. One would expect this result to be robust to the introduction of an adverse selection problem that takes place before moral hazard. Propositions 1 to 3 show that this is true only if effort and type enter as substitutes in the conditional probability of success.

The intuition is as follows. Under a liability restriction on the transfers to the agent ($\alpha = 0$), the countervailing incentives are exactly offsetting so the limited liability constraint is optimally binding for failure. This is not always true under liability restriction on the utility levels of the agent ($\alpha = 1$). In order to increase $p(\theta)$, the principal has to raise $b(\theta)$. But incentive compatibility with respect to type requires (7), i.e. $a'(\theta) \leq 0$ and $b'(\theta) \geq 0$, or equivalently $p'(\theta) \geq -\psi_{12}(p(\theta), \theta)/\psi_{11}(p(\theta), \theta)$. With $b'(\theta) \geq 0$, the principal generates higher $p(\theta)$ from higher types. The impact on $\psi(p(\theta), \theta)$ depends on the production technology parameterization.

First, assume that $f_{12}(e, \theta) = 0$, i.e. with (1), $-\psi_{12}(p(\theta), \theta)/\psi_{11}(p(\theta), \theta) = -\psi_{2}(p(\theta), \theta)/\psi_{1}(p(\theta), \theta)$. Hence, $p'(\theta) \geq -\psi_{2}(p(\theta), \theta)/\psi_{1}(p(\theta), \theta)$ at the optimal contract, so $\psi(p(\theta), \theta)$ increases with $\theta$ or is type independent. The principal trades off the benefits of inducing information revelation from the agent against the cost of generating a higher $\psi(p(\theta), \theta)$ for higher types. She optimally chooses to reduce the generalized agency problem to a moral hazard problem where the optimal level of effort, the optimal sharing rule and
$\psi(p(\theta), \theta)$ are type independent.\footnote{Adverse selection comes into play only to satisfy (7).} She does so because when $f_{12}(e, \theta) = 0$, a higher type has no impact on the marginal productivity of effort. In short, the countervailing incentives are exactly offsetting because of the additive separability of the production technology.

Second, consider the case where $f_{12}(e, \theta) > 0$, i.e. with (1), $-\psi_{12}(p(\theta), \theta)/\psi_{11}(p(\theta), \theta) > -\psi_2(p(\theta), \theta)/\psi_1(p(\theta), \theta)$. If $\alpha = 1$, at the optimal contract, we have $p'(\theta) > -\psi_2(p(\theta), \theta)/\psi_1(p(\theta), \theta)$ for sure, so $\psi(p(\theta)\theta)$ increases with $\theta$. Hence, the incentive for low types to mimic high types remains. In order to increase $p(\theta)$ while preserving the limited liability constraint on utilities, the principal must raise $a(\theta)$, but now this increase will be higher for inefficient types in order to mitigate overstatement of type.

The principal optimally chooses to expand the optimal effort beyond ($FB$) in order to limit the information rent received by the agent added up with the limited liability rent (as illustrated in figure 3\footnote{The expected utility is such that $V^c_u(\theta) = \frac{1}{3} \sqrt{\frac{2}{3} (5 + \theta^2)}$ and the limited liability rent is equal to $p^c_u(\theta)\psi_1(p^c_u(\theta), \theta) = 2\sqrt{\frac{2}{3} \theta^2}$.}). So, if $f_{12}(e, \theta) > 0$, introducing a stricter limited liability rule gives rise to allocative distortions that are rather different from those that prevail under pure moral hazard models.

Figure 3: Expected utility at the solution to $[P]$ in the strong interaction case.
Appendix

A. Properties of $\psi(\rho, \theta)$  Differentiating the conditional density $\rho = f(e, \theta)$ gives $d\rho = f_1(e, \theta)de + f_2(e, \theta)d\theta$. So $de = (1/f_1(e, \theta))d\rho - (f_2(e, \theta)/f_1(e, \theta))d\theta$. With $e = g(\rho, \theta)$, we have $de = g_1(\rho, \theta)d\rho + g_2(\rho, \theta)d\theta$. It follows that $g_1(\rho, \theta) = 1/f_1(g(\rho, \theta), \theta) > 0$ and $g_2(\rho, \theta) = -f_2(g(\rho, \theta), \theta)/f_1(g(\rho, \theta), \theta) < 0$. Then it is straightforward to check that

\[
\begin{align*}
 g_{12}(\rho, \theta) &= \frac{f_{11}(g(\rho, \theta), \theta)f_2(g(\rho, \theta), \theta) - f_{12}(g(\rho, \theta), \theta)f_1(g(\rho, \theta), \theta)}{f_1(g(\rho, \theta), \theta)^3} \leq 0 \\
 g_{11}(\rho, \theta) &= -\frac{f_{11}(g(\rho, \theta), \theta)}{f_1(g(\rho, \theta), \theta)^3} \geq 0 \\
 g_{22}(\rho, \theta) &= -\frac{f_{22}(g(\rho, \theta), \theta)f_1(g(\rho, \theta), \theta) - f_2(g(\rho, \theta), \theta)f_{12}(g(\rho, \theta), \theta)}{f_1(g(\rho, \theta), \theta)^2} \geq 0,
\end{align*}
\]

with $g_{12}(\rho, \theta) = g_{21}(\rho, \theta)$. It follows that

\[
\begin{align*}
 \psi_2(\rho, \theta) &= \varphi'(g(\rho, \theta))g_2(\rho, \theta) < 0 \\
 \psi_1(\rho, \theta) &= \varphi'(g(\rho, \theta))g_1(\rho, \theta) > 0 \\
 \psi_{12}(\rho, \theta) &= \varphi''(g(\rho, \theta))g_1(\rho, \theta)g_2(\rho, \theta) + \varphi'(g(\rho, \theta))g_{12}(\rho, \theta) < 0 \\
 \psi_{11}(\rho, \theta) &= \varphi''(g(\rho, \theta))g_1(\rho, \theta)^2 + \varphi'(g(\rho, \theta))g_{11}(\rho, \theta) > 0.
\end{align*}
\]

Finally, we have

\[
\frac{\psi_2(\rho, \theta)}{\psi_1(\rho, \theta)} \geq \frac{\psi_{12}(\rho, \theta)}{\psi_{11}(\rho, \theta)} \iff \frac{\partial}{\partial \rho} \left( \frac{\psi_1(\rho, \theta)}{\psi_2(\rho, \theta)} \right) \geq 0 \\
\iff \frac{\partial}{\partial \rho} \left( \frac{g_1(\rho, \theta)}{g_2(\rho, \theta)} \right) \geq 0 \\
\iff \frac{f_{12}(g(\rho, \theta), \theta)}{f_1(g(\rho, \theta), \theta)f_2(g(\rho, \theta), \theta)^2} \geq 0. \tag{A.1}
\]

This proves the Properties of $\psi(\rho, \theta)$.

B. Proof of lemma 1  With (3) and the Mirrlees-Rogerson conditions, $p(\hat{\theta}, \theta)$ is such that $b(\hat{\theta}) = \psi_1(p(\hat{\theta}, \theta), \theta)$, so $\psi_{11}(p(\hat{\theta}, \theta), \theta)p_2(\hat{\theta}, \theta) + \psi_{12}(p(\hat{\theta}, \theta), \theta) = 0$ because $b(.)$ does not depend on the true type $\theta$. Both equations must be true at $\hat{\theta} = \theta$; this gives $p_2(\theta, \theta)$ such that

\[
\psi_{11}(p(\theta, \theta), \theta)p_2(\theta, \theta) + \psi_{12}(p(\theta, \theta), \theta) = 0. \tag{B.1}
\]
and expression (5). Hence, \((GIC)\) can be formulated as the pure adverse selection incentive compatibility constraints: \(V(\theta) \geq U(\hat{\theta}, \theta)\). It can be written as \(\theta \in \arg\max_{\theta \in \Theta} U(\hat{\theta}, \theta)\).

It implies the local first-order condition \(U_1(\theta, \theta) = 0\), i.e. \(a'(\theta) + p_1(\theta, \theta)b(\theta) + p(\theta, \theta)b'(\theta) - p_1(\theta, \theta)\psi_1(p(\theta, \theta), \theta) = 0\), or, with (5)

\[
a'(\theta) + p(\theta, \theta)b'(\theta) = 0. \tag{B.2}
\]

Using (B.1) and (B.2), totally differentiating \(U(\theta, \theta)\) reduces to \(V' = -\psi_2(p(\theta, \theta), \theta)\), i.e. expression (6).

It is also necessary to satisfy the local second-order condition \(U_{11}(\theta, \theta) \leq 0\). But differentiating (B.2), it can be written as \(U_{12}(\theta, \theta) \geq 0\), i.e.

\[
p_2(\theta, \theta)b'(\theta) \geq 0.
\]

Plugging in \(p_2(\theta, \theta)\) defined in (B.1), the local second-order condition can be shown to be

\[
b'(\theta) \left( -\frac{\psi_{12}(p(\theta, \theta), \theta)}{\psi_{11}(p(\theta, \theta), \theta)} \right) \geq 0. \tag{B.3}
\]

For reducing (B.3) to \(b'(\theta) \geq 0\) we define the Spence-Mirrlees property. Let \(U^\theta(a, b)\) define the agent’s indifference curves in the contract space \((a, b)\). The negative slope of the indifference curves \(\left(\frac{da}{db}\right)_{U^\theta}\) increases with the agent’s true type since

\[
\frac{\partial}{\partial \theta} \left( \frac{U_2^\theta(a, b)(\hat{\theta}, \theta)}{U_1^\theta(a, b)(\hat{\theta}, \theta)} \right)_{\hat{\theta} = \theta} = p_2(\theta, \theta), \tag{SM}
\]

with \(p_2(\theta, \theta)\) defined as in (B.1). Hence, (SM) can be written as \(-\psi_{12}(p, \theta)/\psi_{11}(p, \theta) > 0\). Thus, (B.3) boils down to \(b'(\theta) \geq 0\).

To show that the allocation is implementable if \(b'(\theta) \geq 0\), assume that the result does not hold, and so there exists some \(\hat{\theta} \neq \theta\) such that \(U(\hat{\theta}, \theta) < U(\hat{\theta}, \theta)\). This is the same as requiring that \(\int_\theta^{\hat{\theta}} U_1(t, \theta)dt > 0\). We know that \(U_1(t, \theta) = a'(t) + p_1(t, \theta)b(t) + p(\theta, \theta)b(t) - p_1(t, \theta)\psi_1(p(t, \theta), \theta)\). But, with (5), we have \(b(t) = \psi_1(p(t, \theta), \theta)\), so \(U_1(t, \theta) = a'(t) + p(t, \theta)b(t)\). Moreover, from (B.2), we can write \(a'(t) = -p(t, t)b'(t)\). Hence, the inequality \(\int_\theta^{\hat{\theta}} U_1(t, \theta)dt > 0\) is equivalent to

\[
\int_\theta^{\hat{\theta}} \{b'(t) (p(t, \theta) - p(t, t))\} dt > 0. \tag{B.4}
\]
Assume $\hat{\theta} > \theta$. With $p_2(\theta, \theta) > 0$ $(SM)$, then $p(t, \theta) - p(t, t) < 0$ since $t > \theta$. This implies that the integral in (B.4) is not positive when $b'(\theta) \geq 0$. Assume that $\hat{\theta} > \theta$. With $(SM)$, then $p(t, \theta) - p(t, t) > 0$ since $t < \theta$. This implies that the function inside the integral in (B.4) is non-negative when $b'(t) \geq 0$. Assume that $\hat{\theta} < \theta$. With $(SM)$, then $p(t, \theta) - p(t, t) > 0$ since $t < \theta$. This implies that the function inside the integral in (B.4) is non-negative when $b'(t) \geq 0$, so the integral in (B.4) is non-positive. Hence, in the presence of $p_2(\theta, \theta) > 0$ $(SM)$, the allocation is implementable if and only if the bonus $b(\theta)$ is non-decreasing in type. Notice that with (B.2) $b'(\theta) \geq 0$ is equivalent to $a'(\theta) \leq 0$. With (5), $b'(\theta) \geq 0$ is equivalent to $p'(\theta) \geq -\psi_12(p(\theta), \theta)/\psi_11(p(\theta), \theta)$. This proves lemma 1.

C. Proof of lemma 2 The agent’s limited liability constraint $l(\theta)$ is non-increasing if and only if, with (9)

$$p'(\theta) \geq -\frac{\alpha \psi_2(p(\theta), \theta) + p(\theta)\psi_12(p(\theta), \theta)}{\alpha \psi_1(p(\theta), \theta) + p(\theta)\psi_11(p(\theta), \theta)}.$$  \hspace{1cm} (C.1)

But we know from the Properties of $\psi$ that

$$\frac{\psi_2(p(\theta), \theta)}{\psi_1(p(\theta), \theta)} \geq \frac{\psi_12(p(\theta), \theta)}{\psi_11(p(\theta), \theta)} \iff -\frac{\psi_12(p(\theta), \theta)}{\psi_11(p(\theta), \theta)} \geq -\frac{\alpha \psi_2(p(\theta), \theta) + p(\theta)\psi_12(p(\theta), \theta)}{\alpha \psi_1(p(\theta), \theta) + p(\theta)\psi_11(p(\theta), \theta)}.$$  \hspace{1cm} (C.2)

Expression (C.2) is a strict inequality when $f_{12}(e, \theta) > 0$ and $\alpha = 1$, otherwise, it becomes an identity. So sufficient condition (7) proves lemma 2.

D. Proofs of propositions 1 to 3 For the sake of simplicity in notation, we omit here the arguments of functions when this causes no ambiguity. Let $y = p'$. We consider the optimal control problem [P] where $y$ is the control and $p$ and $l$ are the states. We associate the adjoint variables $\eta$ and $\lambda$ respectively with $p$ and $l$. We denote by $\mu$ the Kuhn-Tucker multiplier for constraint (8). \hspace{1cm} (15) The Hamiltonian for problem [P] is then

$$\mathcal{H} = (p(\bar{x} - x) + x - l - \psi_1 - \alpha \psi) h - \lambda (\alpha \psi_2 + \psi_12 + y(\alpha \psi_1 + \psi_11)) + \eta y,$$

\hspace{1cm} (15)See Seierstad and Sydsæter (1987).
and its Lagrangian $L = \mathcal{H} + \mu \mathcal{L}$. The maximum principle yields\(^{16}\)

$$\frac{\partial L}{\partial y} = -\lambda (\alpha \psi_1 + p\psi_{11}) + \eta = 0$$  \hspace{1cm} (D.1)

$$\eta' = -\frac{\partial L}{\partial p} = -\left(\pi - \bar{x} - \psi_1(1 + \alpha) - p\psi_{11}\right)h$$  \hspace{1cm} (D.2)

$$+ \lambda (\psi_{12}(1 + \alpha) + p\psi_{112} + y(\psi_1(1 + \alpha) + p\psi_{111}))$$

$$\lambda' = -\frac{\partial L}{\partial l} = h - \mu$$  \hspace{1cm} (D.3)

$$\eta(\theta) = \eta(\bar{\theta}) = 0$$  \hspace{1cm} (D.4)

$$\lambda(\theta)\ell(\theta) = 0, \quad \lambda(\theta) \leq 0, \quad \lambda(\bar{\theta}) \geq 0$$  \hspace{1cm} (D.5)

$$\mu_l = 0, \quad \mu \geq 0, \quad l \geq 0.$$  \hspace{1cm} (D.6)

Notice that differentiating (D.1) with respect to $\theta$ and plugging in the expression derived for $\eta'$ from (D.2) gives the following differential equation for $\lambda$

$$\lambda' = \left(1 - \frac{\pi - \bar{x} - \psi_1}{\alpha \psi_1 + p\psi_{11}}\right)h + \lambda \left(\frac{\psi_{12}}{\alpha \psi_1 + p\psi_{11}}\right).$$  \hspace{1cm} (D.7)

Since $\mathcal{H}$ is linear in $(l, y)$, conditions (D.1) to (D.6) are also sufficient if $\mathcal{H}_{pp} < 0$, i.e. if

$$((2 + \alpha)\psi_{11} + p\psi_{111})h$$

$$> -\lambda [(2 + \alpha)(\psi_{112} + y\psi_{111}) + p(\psi_{1112} + y\psi_{1111})].$$  \hspace{1cm} (D.8)

Let us try to guess the optimal solution, and then verify its optimality by checking that all conditions are satisfied. It is straightforward from (A.1), (C.1) and (C.2) that if sufficient condition (7) is satisfied, then $f_{12} > 0$ and $\alpha = 1$ implies $l' < 0$. Its contrapositive is true, i.e. under (7), $l' = 0$ implies $f_{12} = 0$, $\alpha = 0$, or both. Hence, if $f_{12} > 0$ and $\alpha = 1$, we know that $l' < 0$ and we guess that (7) is inactive; otherwise, we guess that $l = 0$ in which case (7) is active. Therefore, we first gather the proofs of propositions 1 and 2, and then we provide the proof of proposition 3.

\(^{16}\)Transversality conditions (D.4) implicitly assume that $p \in [0; 1]$ for $\theta \in [\bar{\theta}, \bar{\theta}]$, that is $p$ is free at both extremes. This requires $\pi - \bar{x}$ to be well defined.
Proof of propositions 1 and 2 \((f_{12} = 0, \alpha = 0, or both)\). First, consider the cases where \(f_{12} = 0, \forall \alpha \in \{0,1\}\). Assume that \(l^* = 0\) for all \(\theta \in \Theta\), so (7) is active. From (A.1), (C.1) and (C.2), we get

\[
y^* = -\frac{\alpha \psi_2 + p \psi_{12}}{\alpha \psi_1 + p \psi_{11}} = -\frac{\psi_2}{\psi_1} = f_2. \tag{D.9}
\]

Let \(f(e,\theta) = u(e) + v(\theta)\) denote \(\rho\) when \(f_{12} = 0\). Hence, \(y^* = v'\), so \(u(e^*(\theta)) = k\), a constant to be determined. Let us simplify (D.2). From (D.9), we know that

\[
\psi_{12}(1 + \alpha) + p \psi_{112} + y^*(\psi_{11}(1 + \alpha) + p \psi_{111}) \\
= \psi_{12}(1 + \alpha) + p \psi_{112} - \frac{\alpha \psi_2 + p \psi_{12}}{\alpha \psi_1 + p \psi_{11}}(\psi_{11}(1 + \alpha) + p \psi_{111}) \\
= (\alpha \psi_1 + p \psi_{11}) \frac{\partial}{\partial p} \left( \frac{\alpha \psi_2 + p \psi_{12}}{\alpha \psi_1 + p \psi_{11}} \right) \\
= -(\alpha \psi_1 + p \psi_{11}) f_{12} = 0.
\]

So (D.2) is equivalent to \(\eta' = -(\bar{x} - \bar{x} - \psi_1(1 + \alpha) - p \psi_{11}) h\). From (D.4), it follows that \(u(e^*(\theta)) = k\) is such that, \(\forall \theta \in \Theta\)

\[
\int_0^\theta (\bar{x} - \bar{x} - \psi_1(p^*(\theta), \theta)(1 + \alpha) - p^*(\theta) \psi_{11}(p^*(\theta), \theta)) h(\theta) d\theta = 0. \tag{D.10}
\]

At the optimal contract, \(\psi_1\) does not vary with \(\theta\) since with (D.9), \(\psi_{11} y^* + \psi_{12} = 0\). Hence, (D.10) is equivalent to

\[
\bar{x} - \bar{x} - \psi_1 = \alpha \psi_1 + \int_0^\theta p^* \psi_{11} h d\theta > 0,
\]

so \(p^* < p^*\) for all \(\theta \in \Theta\).

Now, let us prove that \(l^* = 0\) for all \(\theta \in \Theta\). If at the solution to (D.10) \(\lambda' < h\), i.e. from (D.3) \(\mu > 0\), then (D.10) is optimal. But the optimal contract requires \(l' \leq 0\) and \(l \geq 0\), so if \(l^* = 0\) is optimal at \(\theta\), then it is optimal for all \(\theta \in \Theta\). Hence, let us show that \(\lambda'(< h(\theta)\) at the solution to (D.10). Using (D.4), \(\eta(\theta) = 0\) so with (D.1), \(\lambda(\theta) = 0\). Hence, at \(\theta = \theta\), equation (D.7) can be written as

\[
\lambda' = \left(1 - \frac{\bar{x} - \bar{x} - \psi_1}{\alpha \psi_1 + p \psi_{11}} \right) h.
\]

It follows that \(\lambda'(\theta) < h(\theta)\) since it is readily shown that \(\bar{x} - \bar{x} - \psi_1 > 0\) at the solution to (D.10). Therefore, (D.10) is optimal.
With (D.9), the neglected constraint (7) is trivially satisfied since it is binding. Finally, notice that a sufficient condition for the concavity of \( H \) is \( \psi_{111} \geq 0 \) since the right hand side of (D.8) is equal to zero when \( f_{12} = 0 \). Indeed, with (D.9),

\[
-\frac{\partial}{\partial p} \left( \frac{\psi_{12}}{\psi_{11}} \right) = f_{12} = 0 \Rightarrow -\frac{\psi_{12}}{\psi_{11}} = -\frac{\psi_{112}}{\psi_{111}} = \psi_{1111} = y^s.
\]

We assume that \( f_{111} \) is not too big to get \( \psi_{111} \geq 0 \). This proves proposition 1.

Second, consider the case where \( f_{12} > 0 \) and \( \alpha = 0 \). Assume that \( l^c_\theta = 0 \) for all \( \theta \in \Theta \), so (7) is active. From (C.1), it follows that \( y^c_\theta = -\psi_{12}/\psi_{11} \). Now, consider the solution to \([P-MH]\).

Totally differentiating \((MH)\) yields

\[
y^c_{MH} = -\frac{\psi_{12} + p\psi_{112}}{2\psi_{11} + p\psi_{111}}.
\]

The pure moral hazard level of effort is implemented despite the nonobservability of type, i.e. \( p^c_{MH} = p^c_\theta \) if and only if

\[
\frac{-\psi_{12} + \rho \psi_{112}}{2\psi_{11} + \rho \psi_{111}} = -\frac{\psi_{12}}{\psi_{11}} \quad \Rightarrow \quad -\psi_{11}\psi_{12} + \rho \psi_{12}\psi_{111} - \rho \psi_{11}\psi_{112} = 0
\]

\[
\Leftrightarrow \quad -\rho \psi_{11} \left( \frac{\partial}{\partial \rho} \left( \frac{\psi_{12}}{\psi_{11}} \right) \right) = 1
\]

i.e. if and only if \( \psi_{12}/\psi_{11} \) is unit elastic with respect to \( \rho \). This proves proposition 2.

**Proof of proposition 3** \((f_{12} > 0 \text{ and } \alpha = 1)\). Let \( J(\theta) = H(\theta)/h(\theta) \). Assume constraint (7) is inactive. It is readily shown that \( l^c_\theta < 0 \) and so (8) is binding only at \( \bar{\theta} \). From (D.1) and (D.4), we have \( \lambda(\bar{\theta}) = \lambda(\bar{\theta}) = 0 \). Using \( \lambda(\bar{\theta}) = 0 \) and (D.3), we write \( \lambda(\bar{\theta}) - \lambda = \int_{\theta}^{\bar{\theta}} (h - \mu) dt \).

So \( \lambda = H - 1 + M \), with \( M = \int_{\theta}^{\bar{\theta}} \mu dt \). Using \( \lambda(\bar{\theta}) = 0 \), we get \( M(\bar{\theta}) = 1 \). Moreover, since (8) is slack for \( \theta < \bar{\theta} \), we have \( \mu = 0 \). Hence \( M' = -\mu = 0 \). So \( M \) is a constant equal to 1 and \( \lambda = H \). From (D.7), we obtain \( p^c_\theta(\theta) \) such that, \( \forall \theta \in \Theta \)

\[
\bar{x} - \bar{x} = \psi_1(p^c_\theta(\theta), \theta) + J(\theta)\psi_{12}(p^c_\theta(\theta), \theta).
\]

With \( J(\bar{\theta}) = 0 \), \( p^c_\theta(\theta) \) is such that there is no distortion for the least efficient type and an upward distortion for all other types.
Moreover, totally differentiating condition (D.11) yields

\[ y^c_u = -\frac{\psi_{12}(1 + J') + J\psi_{122}}{\psi_{11} + J\psi_{112}}. \]

Constraint (7) is inactive if and only if \( y^c_u > -\psi_{12}/\psi_{11} \), i.e., after rearrangement,

\[ \psi_{11}\psi_{12}J' + (\psi_{11}\psi_{122} - \psi_{12}\psi_{112})J < 0. \]

Finally, notice that sufficient conditions for the concavity of \( H \) are \( \psi_{11} + J\psi_{112} > 0 \) and \( \psi_{111} + J\psi_{1112} \geq 0 \). Indeed, with \( f_{12} > 0 \), the solution entails \( \lambda = H \), so the inequality (D.8) can be written as \( 3(\psi_{11} + J\psi_{112}) + p(\psi_{111} + J\psi_{1112}) + Jy^c_u\psi_{111}(3 + p) > 0 \). We assume that these sufficient conditions hold. This proves proposition 3. \( \square \)

**References**


